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Quantum Matrices in N Dimensions

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We obtain the commutation relation between rows and columns of $N \times N$ quantum matrices. We derive the quantum determinant and discuss its property in terms of the q-deformed Levi-Civita symbol. We find the inverse and trace of $N \times N$ quantum matrices. Finally we discuss the q-deformed complexification of the quantum matrices.

Since Manin (1987, 1988, 1989) proposed the idea of the quantum plane, much has been developed in this direction. The application of noncommutative differential geometry to the quantum matrix group was made by Woronowicz (1987, 1989). Wess and Zumino (1990; Zumino, 1991) considered one of the simplest examples of noncommutative differential calculus over Manin's plane. They developed a differential calculus over the quantum hyperplane covariant with respect to the action of the quantum deformation of $GL(n)$ which is called $GL_q(n)$. Following these results, much work has been done in this direction (Schmidke *et al.*, 1989; Schirrmacher, 1991*a,b*; Schirrmacher *et al.*, 1991; Burdik and Hlavaty, 1991; Hlavaty, 1991; Burdik and Hellinger, 1992; Ubriaco, 1992; Giler *et al.*, 1991). Some recent papers (Ewen *et al.*, 1991; Ge *et al.*, 1992; Kupershmidt, 1992) cover the 2×2 quantum matrix and its properties. The main purpose of this paper is to deal with the $N \times N$ quantum matrix and its properties. First we set up commutation relations between rows and columns of the $N \times N$ quantum matrix, and we use the q-deformed Levi-Civita symbol and its useful property (Chung *et al.*, 1993) to define the quantum

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determinant and inverse of the quantum matrix. Finally we define the q-deformed trace of the quantum matrix and prove that it remains invariant under the similarity transformation.

The N-dimensional quantum space is realized by N noncommuting variables satisfying the following rule:

$$x_i x_j = q x_j x_i \qquad (i < j) \tag{1}$$

Consider the linear transformation of these variables,

$$x'_i = a_{ij} x_j \tag{2}$$

where the a_{ij} are entries of the quantum matrix A commuting with the x_i. If we demand that the relation (1) holds in the transformed coordinate, we have

$$x'_i x'_j = q x'_j x'_i \qquad (i < j) \tag{3}$$

Inserting equation (2) into equation (3) gives

$$a_{il} a_{jl} = q a_{jl} a_{il} \qquad (i < j) \tag{4}$$

$$a_{ik} a_{jl} + q^{-1} a_{il} a_{jk} = a_{jl} a_{ik} + q a_{jk} a_{il} \qquad (i < j, k < l) \tag{5}$$

On the other hand, the differential forms of these variables should satisfy

$$dx_i\, dx_j = -\frac{1}{q}\, dx_j\, dx_i \qquad (i < j) \tag{6}$$

$$dx_i^2 = 0 \tag{7}$$

We demand that the relations (6) and (7) also hold in the transformed coordinate. From the relation

$$dx_i'^2 = 0 \tag{8}$$

we have

$$a_{ik} a_{il} = q a_{il} a_{ik} \qquad (k < l) \tag{9}$$

From the relation

$$dx'_i\, dx'_j = -\frac{1}{q}\, dx'_j\, dx'_i \qquad (i < j) \tag{10}$$

we obtain

$$a_{ik} a_{jl} - q a_{il} a_{jk} = -\frac{1}{q} a_{jk} a_{il} + a_{jl} a_{ik} \qquad (i < j, k < l) \tag{11}$$

From equations (5) and (11) we obtain

$$a_{il}a_{jk} = a_{jk}a_{il} \qquad (i<j, k<l) \tag{12}$$

Therefore, in order for A to be a quantum matrix, the entries of A, a_{ij}, should satisfy the following relations:

$$a_{il}a_{jl} = qa_{jl}a_{il} \qquad (i<j) \tag{13}$$

$$a_{ik}a_{jl} + q^{-1}a_{il}a_{jk} = a_{jl}a_{ik} + qa_{jk}a_{il} \qquad (i<j, k<l) \tag{14}$$

$$a_{il}a_{jk} = a_{jk}a_{il} \qquad (i<j, k<l) \tag{15}$$

$$a_{ik}a_{il} = qa_{il}a_{ik} \qquad (k<l) \tag{16}$$

Now consider the quantum determinant of the $N \times N$ quantum matrix A. Let us denote the quantum determinant of A by $\det_q A = D$, which is determined by the relation

$$dx'_1\, dx'_2 \cdots dx'_N = D\, dx_1\, dx_2 \cdots dx_N \tag{17}$$

Applying the transformation rule (2) to (17), we have

$$D = \sum_{i_1 i_2 \cdots i_n} \epsilon^q_{i_1 i_2 \cdots i_n} a_{1i_1} \cdots a_{Ni_N} \tag{18}$$

Here the q-deformed Levi-Civita symbol $\epsilon^q_{i_1 i_2 \cdots i_n}$ satisfies the following properties:

$$\epsilon^q_{12\cdots N} = 1 \tag{19}$$

$$\epsilon^q_{i_1 \cdots ij \cdots i_N} = (-q)\epsilon^q_{i_1 \cdots ji \cdots i_N} \qquad \text{for} \quad i>j \tag{20}$$

For example, let us consider the $N=3$ case. We have six nonvanishing components of the q-Levi-Civita symbol,

$$\begin{aligned} \epsilon^q_{123} &= 1 \\ \epsilon^q_{213} &= -q \\ \epsilon^q_{132} &= -q \\ \epsilon^q_{312} &= (-q)^2 \\ \epsilon^q_{231} &= (-q)^2 \\ \epsilon^q_{321} &= (-q)^3 \end{aligned} \tag{21}$$

Therefore the quantum determinant of the 3×3 quantum matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \tag{22}$$

is given by

$$\det_q A = aei + (-q)^2 bfg + (-q)^2 cdh$$
$$+ (-q)^3 ceg + (-q)bdi + (-q)afh$$
$$= a \det_q \begin{pmatrix} e & f \\ h & i \end{pmatrix} + (-q)b \det_q \begin{pmatrix} d & f \\ g & i \end{pmatrix} + (-q)^2 c \det_q \begin{pmatrix} d & e \\ g & h \end{pmatrix} \quad (23)$$

This example shows that we can compute the quantum determinant of the $N \times N$ quantum matrix from the N quantum determinants of the $(N-1) \times (N-1)$ quantum matrix, which we call minors. Then we conclude that

$$\det_q A = \sum_{i=1}^{N} (-q)^{i-1} a_{1i} M_{1i} \quad (24)$$

where M_{ij} is called the minor of a_{ij}.

Now we prove that the important property of the determinant still holds in the case of the quantum determinant, which is

$$\det_q (AB) = \det_q A \det_q B \quad (25)$$

The proof of the above property is as follows:

$$\det_q(AB) = \sum_{i_1 i_2 \cdots i_N} \epsilon^q_{i_1 i_2 \cdots i_N} (AB)_{1 i_1} \cdots (AB)_{N i_N}$$
$$= \sum_{i_1 i_2 \cdots i_N} \sum_{k_1 k_2 \cdots k_N} \epsilon^q_{i_1 i_2 \cdots i_N} a_{1 k_1} b_{k_1 i_1} \cdots a_{N k_N} b_{K_N i_N}$$
$$= \sum_{i_1 i_2 \cdots i_N} \sum_{k_1 k_2 \cdots k_N} (a_{1 k_1} \cdots a_{N k_N}) \epsilon^q_{i_1 i_2 \cdots i_N} b_{k_1 i_1} \cdots b_{k_N i_N}$$
$$= \sum_{k_1 k_2 \cdots k_N} (a_{1 k_1} \cdots a_{N k_N}) \epsilon^q_{k_1 k_2 \cdots k_N} \det_q B$$
$$= \det_q A \det_q B \quad (26)$$

where we used the formula

$$\sum_{i_1 i_2 \cdots i_N} \epsilon^q_{i_1 i_2 \cdots i_N} b_{k_1 i_1} \cdots b_{k_N i_N} = \epsilon^q_{k_1 k_2 \cdots k_N} \det_q B \quad (27)$$

Formula (27) can be easily proved from the definition of the q-Levi-Civita symbol and its properties (Chung *et al.*, 1993).

From the definition of the quantum determinant and the properties of the q-Levi-Civita symbol, we can easily prove that the quantum determinant of the quantum matrix commutes with all entries of the quantum matrix, which means that the quantum determinant is a central element. But in the case of multiparameter deformation, it is not the case, because the quantum determinant does not commute with entries of the quantum matrix.

In order for the $N \times N$ quantum matrix to form a Hopf algebra, there should be an antipode satisfying the antipode axiom. In the world of $N \times N$ quantum matrices the antipode is an inverse of the quantum matrix. The inverse of the quantum matrix A is defined by

$$(A^{-1})_{ij} = \frac{1}{|A|_q}(-q)^{i-j}\det_q(M_{ji}) \tag{28}$$

where $|A|_q$ means $\det_q A$. It is straightforward to prove that

$$(AA^{-1})_{ij} = \delta_{ij} \tag{29}$$

The left-hand side of equation (29) is

$$\begin{aligned}(AA^{-1})_{ij} &= \sum_k a_{ik}a_{kj}^{-1} \\ &= \sum_k a_{ik}\frac{1}{|A|_q}(-q)^{k-j}\det_q(M_{jk}) \\ &= \frac{1}{|A|_q}\sum_k(-q)^{k-j}a_{ik}\det_q(M_{jk}) \\ &= \delta_{ij}\end{aligned} \tag{30}$$

and equation (29) is proved.

It is very easy to prove that the inverse operation obeys the antipode axiom, which means that

$$(AB)^{-1} = B^{-1}A^{-1} \tag{31}$$

where A and B are arbitrary quantum matrices.

Now we use the information on the inverse and quantum determinant to define the quantum trace of the quantum matrix A as

$$\mathrm{Tr}_q(A) = \sum_{i=1}^{N} q^{2(i-1)}a_{ii} \tag{32}$$

Then we have

$$\mathrm{Tr}_q(BAB^{-1}) = \sum_{ijk} q^{2(i-1)}b_{ij}a_{jk}b_{ki}^{-1} \tag{33}$$

Using the definition of the inverse of the quantum matrix, we get

$$\begin{aligned}\mathrm{Tr}_q(BAB^{-1}) &= \frac{1}{|B|_q}\sum_{ijk} q^{2(i-1)}(-q)^{k-i} b_{ij}|M_{ik}|_q a_{jk} \\ &= \frac{1}{|B|_q}\sum_{jk} q^{2(k-1)}\delta_{jk}|B|_q a_{jk} \\ &= \sum_j q^{2(j-1)} a_{jj} = \mathrm{Tr}_q(A) \end{aligned} \tag{34}$$

where M_{ik} is a minor of b_{ik} and B is another quantum matrix whose entries commute with those of A. Therefore we conclude that the quantum trace remains invariant under the similarity transformation.

Now we consider the q-deformed complexification of the quantum matrix. There exist four different types of complex conjugation of N noncommuting variables:

I. $x_i = x_{N+1-i}$ $(i = 1, 2, \ldots, N)$.
II. $\bar{x}_i = x_i$ $(i = 1, 2, \ldots, N)$.
III. $x_i\bar{x}_i = 1$ (not summed) $(i = 1, 2, \ldots, N)$.
IV. $x_i\bar{x}_i = 1$ (not summed) $(i = 1, 2, \ldots, m)$.

$$x_\alpha = \bar{x}_\alpha \qquad (\alpha = m+1, \ldots, N)$$

Then we have the relations between a_{ij} and $\bar{a}_{ij}$, the complex conjugation of a_{ij}, in addition to the relations (13)–(16), which are written as follows:

I. $\bar{a}_{i,N+1-j} = a_{i,N+1-j}$ $(i, j = 1, 2, \ldots, N)$.
II. $\bar{a}_{ij} = a_{ij}$ $(i, j = 1, 2, \ldots, N)$.
III. $a_{ij} = a_{ii}\delta_{ij}$, $a_{ii}\bar{a}_{ii} = 1$ $(i, j = 1, 2, \ldots, N)$.
IV. $a_{ij} = a_{ii}\delta_{ij}$, $a_{ii}\bar{a}_{ii} = 1$ $(i, j = 1, 2, \ldots, N)$.

$$a_{\alpha\beta} = \bar{a}_{\alpha\beta} \qquad (i, j = 1, 2, \ldots, m, \alpha, \beta = m+1, \ldots, N)$$

Although this paper is more or less mathematical, we think that this information on $N \times N$ quantum matrices will enable one to treat the q-deformed quantum mechanics in N-dimensional quantum space-time. We hope that these properties of $N \times N$ quantum matrices will be applied to problems in theoretical physics. There remain some unsolved problems relating to the topics of this paper. One is to define a q-orthogonal, q-Hermite, and q-unitary matrix and another is to construct a q-deformed matrix quantum mechanics by defining the q-ket and q-eigenvalue problem, etc. We think that these problems and related topics will be clear in the near future.

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